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# A New Look at Linear Viscoelasticity

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**Abstract**

A novel graphical representation of a viscoelastic behaviour is derived using a model which includes a stochastic transfer of strain energy into dissipative processes. The picture of a mechanical behaviour thus obtained offers a different look at a material from that found in empirical plots of force and deformation. By precisely matching the stochastic model to the behaviour of the standard linear viscoelastic solid, the derived graphical output also adopts a characteristic linearity.

*Key words:* Visco-elasticity; Stress relaxation; Modelling; Strain energy; Energy dissipation, Enet model

## 1. Introduction

The mechanical behaviour of a material is implicitly determined its ability to store or dissipate energy applied during a deformation. Linear elastic springs and viscous dashpots are fundamental components which can be assembled to model these processes of energy storage and dissipation and thus simulate a mechanical behaviour for a viscoelastic material.

Combinations of springs and dashpots have been used to simulate specific mechanical behaviours and linear viscoelastic theory has been widely developed and utilised in a broad range of rheological applications (Ferry (1970), Fung (1977)).

Here an alternative construct, which will be referred to as the enet model, is presented which differs from linear viscoelasticity in one important aspect. In linear viscoelasticity the dashpots dissipate the strain energy in serially connected springs as a continuous process. In the enet model, this dissipation occurs only at critical points of energy input as defined by failure criteria of enet elements which replace the dashpots in the mechanical network.

A highly specific form of the enet model has been found to match precisely the mechanical properties of the standard linear viscoelastic solid. This enables their relationship to be explored.

Furthermore, the enet model enables a new graphical template to be created which gives a visualisation of the 'strategy' the material deploys to handle applied mechanical energy. This template may give a deeper insight into the fundamental processes of energy storage and dissipation than is apparent in empirical mechanical force-deformation data and it is hoped that the technique may provide a useful new tool to examine structure-function relationships in materials.

As a first step the technique has been used to investigate the mechanical behaviour of the standard linear viscoelastic solid and its characteristic linearity is clearly evident in a graphical template obtained.

## 2. Standard linear viscoelastic solid

The standard linear viscoelastic solid shown in Fig. 1 is an elementary combination of a spring with stiffness $k_e$ in parallel with a series combination of a spring with stiffness $k_v$ and a dashpot with a co-efficient of viscosity $c_v$ . It provides the most general linear viscoelastic relationship to include load, deformation and their first derivatives (Fung (1977)).

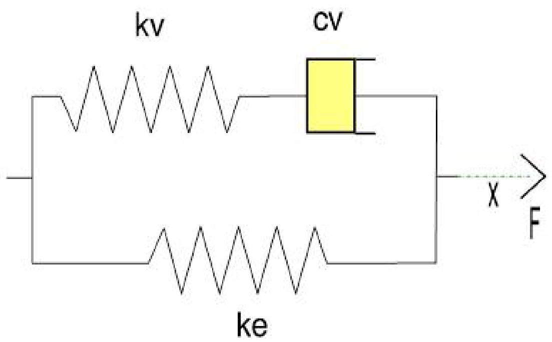


Fig. 1. The combination of springs and dashpots which comprise the standard linear viscoelastic solid.

When the standard linear viscoelastic solid is loaded, both springs contribute to the force-deformation behaviour. With time, the strain energy in spring $k_v$ will be dissipated through its connecting dashpot, so that the force contributed by this arm of the standard solid will decay. Ultimately, the resulting force will be that due only to spring $k_e$ and this is termed the 'equilibrium force'.

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When the standard linear viscoelastic solid is subject to a constant rate of extension, the concept of superposition can be employed to calculate the force *F* necessary to reach a deformation *x* at time *t* thus :-

$$F = k_e . x + \frac{k_v . x . \tau}{t} . \left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \qquad (1)$$

where $\tau = \frac{c_v}{k_v}$

Eqn. 1 provides the total force-deformation ($F$ - $x$) characteristics and equilibrium force-deformation characteristics ($F_{eq}$ - $x$) for the standard linear viscoelastic solid, where $F_{eq} = k_e . x$

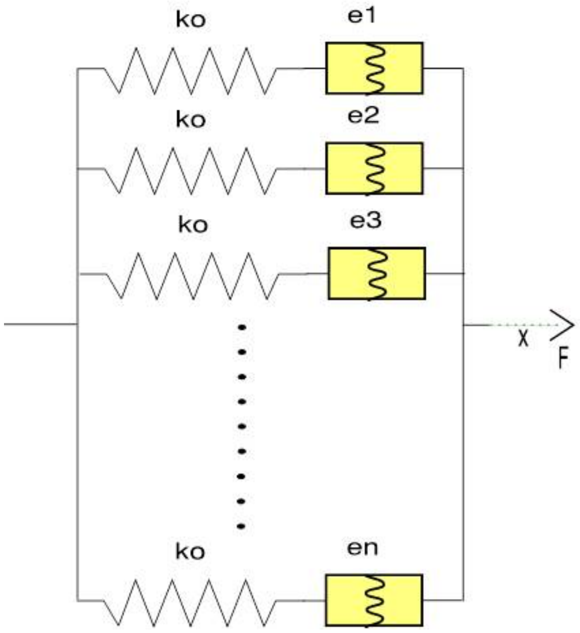


Fig. 2. A generalised parallel assembly of spring-enet mechanical units in which each spring has a constant stiffness $k_o$ so that the applied strain energy will be distributed evenly between the spring elements. The individual enet elements will fail when the energy stored in their connecting spring reaches a critical value $e_1$ , $e_2$ , ......... , $e_n$ (for an initial assembly $n = p_i$). Therefore, as the parallel assembly begins to be loaded, some of the fundamental units will fail earlier than others, ceasing then to be "active". On subsequent loading of the assembly, the additional applied energy will be distributed only to those remaining active fundamental units, which will still contribute to the elastic behaviour. Those units that have failed will temporarily contribute their force to the total viscous force, until this contribution decays to zero.

## 3. Enet model

The fundamental mechanical unit of the enet model comprises an elastic spring connected in series with an enet element which remains dormant mechanically until the deformation in the connected spring reaches a critical value at which point mechanical failure occurs. The enet model is constructed as a parallel array of $p_i$ fundamental mechanical units in which the springs each have the same stiffness $k_0$ and a variable enet failure deformation (Fig. 2).

The proportion of enet elements in the parallel assembly which fail may be calculated as a function of the applied deformation or strain energy. Later a specific pattern of enet failure will be described which enables the model to match the mechanical behaviour of the standard linear viscoelastic solid.

An individual enet failure is associated with two subsequent events. The force in the connected spring then decays slowly in a manner comparable with the equivalent stress relaxation of the standard linear viscoelastic solid. That is, after enet failure:-

$$F_v = F_{fl} . \exp\left(\frac{-(t - t_{fl})}{\tau}\right) \qquad (2)$$

where $F_v$ is the subsequent contribution made to the viscous force at time *t* by the failed fundamental unit, $F_{fl}$ was the maximum force occurring at time $t_{fl}$ when the unit failed.

Also in response to a spring-enet unit failure, a variable number $q_s$ additional fundamental units are recruited into a load carrying capacity. These recruited units amalgamate into additional parallel assemblies, which behave as their precursors as the deformation is continued.

This sequence of fundamental unit failure and recruitment is suitable for an iterative algorithmic approach. Following this algorithm, with a time increment of $t_r$ , at any time *t* the enet model will then contain $t / t_r$ parallel assemblies of fundamental mechanical units each contributing a viscous force from those units which have failed and an equilibrium force when the enet elements remain connected. A computer model may thus calculate a ($F$ - $x$) and ($F_{eq}$ - $x$) behaviour which can be compared directly to measured properties of a material or to the equivalent behaviour of the standard linear viscoelastic solid as given by Eqn. 1.

## 4. Graphical representation of enet model behaviour

The iterative process of fundamental mechanical unit loading, failure and recruitment can rapidly lead to a numerical model of considerable complexity. It is useful, therefore, to visualise the workings of the spring-enet model graphically and a new template has been devised to do this.

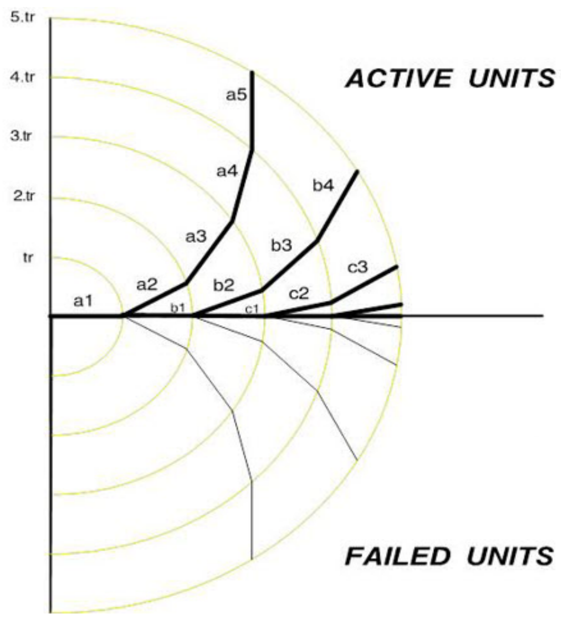


Fig. 3. The discrete time intervals which form the incremental steps of the simulation ($t_r$ , 2. $t_r$, 3. $t_r$, etc.) are shown as concentric circles emanating from the origin of the graph. The angle of lines drawn onto this template indicates the proportion of spring-enet units to have failed; a horizontal indicating no failures have occurred and a vertical line indicative of the complete failure of a particular parallel assembly. The proportion of the initial assembly which fails at the first time increment $t_r$ is thus indicated by the angle of line $a_2$. The recruited second assembly is then represented by $b_1$, and the subsequent failure of parts of the initial and second assemblies of units which occurs at the second time increment 2. $t_r$ are indicated by the angle of lines $a_3$ and $b_2$ respectively while $c_1$ represents the consequent further recruitment of units to form a third parallel assembly. As the simulation advances, further lines will be added as additional assemblies are recruited, and those earlier lines will gradually steepen as more of their constituent units will fail.

In this graphical representation of enet model behaviour concentric circles mark the iterative intervals $t_r$. The failure and recruitment patterns in each parallel assembly of fundamental units are represented by lines drawn onto the template as shown in Fig. 3. The angle of the line specifies the proportion of the assembly which has failed.

Furthermore, the colour of each line in this new type of picture can be used to register the force contributed by the assembly of fundamental units that the line represents. The pattern above the horizontal axis will represent the forces from the active units which together make up the elastic equilibrium behaviour of the model, while below the axis the contribution to the decaying viscous force of the failed fundamental units is recorded.

## 5. Comparison of the linear viscoelastic and enet model

The computed ($F$ - $x$) and ($F_{eq}$ - $x$) behaviour of the enet model has been match to that of the standard linear viscoelastic solid (Eqn. 1) using a simplex-based Gauss least-squares method. Representative values of the model parameters can thus be estimated at the point of best fit. The closeness of fit between the two models has been measured using the square of the co-efficient of correlation $R^2$ (Weatherburn (1968)) on both the ($F$ - $x$) and ($F_{eq}$ - $x$) measurements.

A precise simulation of the linear viscoelastic behaviour has been achieved using an enet model in which the number of fundamental units which fail in each assembly in the time period ($t$ , $t + t_r$) is given by the recursive formula:-

$$\text{number of failures between time } t \text{ and } (t + t_r) = \phi \;.\; n_a(t) \;.\; n_a(0) \qquad (3)$$

where $n_a(t)$ is the number of units in the particular assembly which are active at time $t$, $n_a(0)$ is the initial number of units when that same assembly was first formed and $\phi$ is a constant.

Therefore, the number of units in each parallel assembly which fail during a time interval follows an inductive sequence being dependent only on the initial and current numbers of active fundamental units and not on any applied deformation.

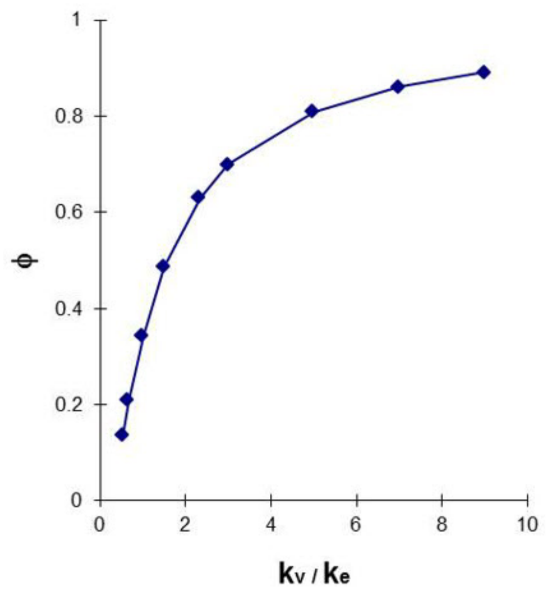


Fig. 4. The matching of the mechanical behaviours of the enet model to the standard linear viscoelastic solid, using the method of least squares, allows the estimation of parameter $\phi$ of Eqn. 3 which corresponds to specific values of $k_v$ / $k_e$ .

As $k_v$ / $k_e$ increases, so does the value of $\phi$, as at each time increment in the numerical enet model, a greater proportion of the fundamental mechanical units need to fail to simulate an increasingly viscous behaviour.

**Table 1**

| Time | Initial Assembly | 2nd Assembly | 3rd Assembly | 4th Assembly |
|---|---|---|---|---|
| 0 | 1 | | | |
| $t_r$ | $1 - \phi$ | $\phi$ | | |
| $2.t_r$ | $(1 - \phi)^2$ | $\phi.(1 - \phi^2)$ | $\phi.( \phi^2 - \phi + 1)$ | |
| $3.t_r$ | $(1 - \phi)^3$ | $\phi.(1 - \phi^2)^2$ | $\phi.( \phi^2 - \phi + 1)\ [1 - \phi^2.( \phi^2 - \phi + 1)]$ | $\phi^7 - 2.\ \phi^6 + 2.\ \phi^5 - 2.\ \phi^4 + 3.\ \phi^3 - 2.\ \phi^2 + \phi$ |
| $4.t_r$ | $(1 - \phi)^4$ | $\phi.(1 - \phi^2)^3$ | $\phi.( \phi^2 - \phi + 1)\ [1 - \phi^2.( \phi^2 - \phi + 1)]^2$ | ******* |

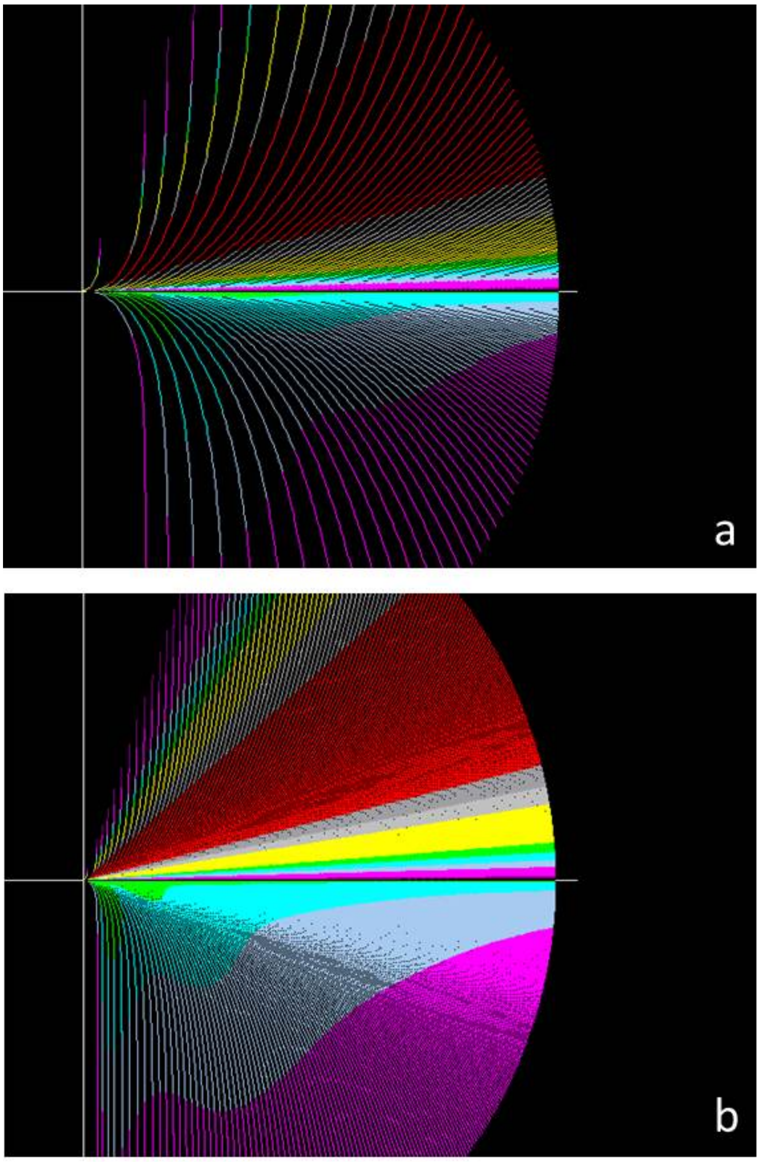


Fig. 5. Here the enet model is configured using Eqn. 3 to replicate the mechanical behaviour of a standard linear viscoelastic solid (with $k_e = k_v = 50$ N.mm$^{-1}$ and $\tau = 10$ s). This enet model was subject to an extension at 1 mm s$^{-1}$ which continued for 10 seconds. The colour changes record the forces in the fundamental units which increase from black to a maximum red level.

In (a) the model time increment $t_r$ is 0.1 s, so that the lines representing the parallel assemblies of fundamental units can be individually traced ($\phi = 0.345$). This time increment is shorter in (b) with $t_r = 0.02$ s. In this case the individual lines blend into a continuous pattern, because of their greater number ($\phi = 0.342$).

## 6. Results

The enet model incorporating the enet element failure criteria given in Eqn. 3 provides a ($F$ - $x$) and ($F_{eq}$ - $x$) behaviour which has been fitted to comparable data from Eqn. 1 for the standard linear viscoelastic solid. As the time interval $t_r$ decreases, this fit improves to a level at which the two models are identical. For values of $t_r < 0.25$ s over a ten second simulation $R^2 > 0.99995$.

At this point of convergence of fit, the estimated model parameter values become equivalent. That is, $p_i$ is such that the initial stiffness of the enet model $p_i.k_0$ equals $k_e + k_v$ , the values of $\tau$ for each model are the same and $q_s$ is made equal to unity in the enet model ensuring a constant number of active fundamental units and a constant stiffness throughout the simulation.

The fourth enet model parameter $\phi$, present in Eqn. 3, defines the level of enet failure and is equivalent to the degree of viscosity in the standard linear solid, which is itself determined by the ratio of the two spring stiffnesses $k_v$ / $k_e$ (Fig. 4).

The rather simple rule in Eqn. 3 does lead quickly to a complex numerical sequence of active spring-enet units in a string of parallel assemblies which develop as the deformation proceeds. This sequence is shown with the simplifying assumption that $p_i$=1 in Table 1.

The equivalent enet graphical template in Fig. 5 presents a more comprehensible pattern. As the enet model is loaded, the lines sweep outward and away from the horizontal, illustrating the gradual failure of fundamental units. The colours in Fig. 5a show the elastic forces in each assembly rise to a maximum and then fall with time as the proportion of units to have failed in the assembly increases.

As the time increment is shortened in Fig. 5b, the lines blend into a more continuous and distinctive pattern. Above the horizontal axis, the colour bands which indicate the contributions made to the elastic equilibrium force radiate outwards, linearly on the circular template. Note the individual lines which comprise this pattern are not linear but cut across the coloured linear boundaries.

At each time increment, that is, along every circular segment drawn above the horizontal axis, the colours retain a constant angular relationship. As the deformation proceeds, there appears to be a constancy in the way the equilibrium force is aggregated from the individual contributions of the active spring-enet units.

## 7. Discussion

The value of any constitutive model is determined by the insight the model provides into the material it is set to represent. The patterns on the graphical template derived here are descriptive of the strategy a material may employ to handle the energy of deformation and the concept may provide a useful tool in the analysis of structure-function relationships in the material.

The scope of the present paper is limited to forging a link between the enet model and the theoretical foundations of linear viscoelastic theory. This is convenient because the behaviour of the standard linear viscoelastic solid is free from noise and any secondary and tertiary influences which combine in creating the mechanical behaviour of a real material. Therefore, a perfect matching of the linear viscoelastic and enet models has proved possible and at this point of perfect fit, the graphical template obtained from the enet model does show a characteristic linearity consistent with its theoretically linear behaviour.

Various combinations of spring and dashpot components have been used to enable linear viscoelastic theory to simulate the mechanical behaviour of specific materials, although a precise representation often requires the extension of this model to infinite arrays of components as defined through the relaxation spectra of linear viscoelasticity (Ferry (1970)). Also, it has been shown that linear viscoelasticity does provide a generalised representation of thermodynamic phenomena occurring within the vicinity of an equilibrium (Biot (1954)). However, non-linear materials, such as polymers and biological tissues, do require nonlinear constitutive formulae to characterise their mechanical properties (Lockett (1972)). Nonlinearities may be introduced into the enet model either by associating a statistical distribution to the failure characteristics of the enet population, or by allowing the recruitment parameter $q_s$ to take a value greater than unity. This latter step may reflect a gradual recruitment of load bearing components within a material micro-structure as is understood to occur when biological tissues are deformed (Egan (1987)).

In linear viscoelasticity the conditions for energy dissipation are narrowly defined by the assumed Newtonian behaviour of the dashpot: that its rate of deformation is linearly proportional to the applied force. This has necessitated the incorporation of complex non-Newtonian dissipative mechanisms into viscoelastic theory (Ferry (1970)).

This tightly restricted behaviour of the standard linear viscoelastic solid is matched by a highly specific definition of enet failure criteria in the matched model given in Eqn. 3. Other formulae and statistical failure criteria (uniform, Gaussian, exponential, Weibull and gamma) have been applied without convergence to the properties of the linear viscoelastic solid, leading the author to believe that Eqn. 3 represents a special and possibly unique solution to achieve this identical matching.

What is surprising is that the failure criteria defined by Eqn. 3 are independent of deformation and include only an inductive numerical sequence to control the enet failure events.

Rowse (1953) used a 'bead-spring' model to characterise interactions which occur along a flexible polymer chain and thereby derive a mechanical behaviour. This stochastic model can similarly be made analogous to a linear viscoelastic behaviour (Ferry (1970), Doi and Edwards (1986)). The enet model uses a similar stochastic approach to characterise the conditions under which strain energy stored within the elastic structures of a loaded material is transferred or dissipated.

This transfer of energy from elastic structures into whatever dissipative mechanism exist within a material must act to reduce the total potential energy in the system. If this energy transfer occurs with a disruption of some interactions within the material micro-structure, then it is likely that such interactions will fail in a stochastic manner when loaded to a critical value (Egan (1987)).

The analytical tools of linear viscoelasticity have parallel applications in other disciplines where linear systems have relevance. Similarly, the techniques discussed here may prove to have a wider applicability.

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